

Supergroup formulation of Plebański action of gravity

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(Dated: June 22, 2010)

Abstract

General relativity can be formulated as a $SU(2)$ BF-theory with constraints, as has been shown, by Plebański. The cosmological constant term can be obtained from the constraint term, following from the consistency of the equations of motion, as recently shown by Krasnov. We consider an $OSp(2|1)$ invariant, supergravity extension of this theory, for which the consistency of the equations of motion and the constraints contribute as well to the cosmological constant terms of Townsend's supergravity. The Kalb-Ramond invariance is shortly discussed.

PACS numbers: 4.65.+e, 4.60.-m, 4.50.Kd

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I. INTRODUCTION

The problem of quantization of gravity has been for many years the focus of activity for many people. The most promising approaches to solve it have been superstring theory and loop quantization [1], the last one based on the Ashtekar hamiltonian formulation. In a similar direction, there are attempts to quantize in a diffeomorphism invariant setting, by means of a graph approach of the path integral formulation of the Plebański action [2], i.e. spin-foam [3] or state-sum models, in which the simplicity constraints impose conditions on the allowed representations of the associated symmetry group [4]. Inversely, starting from spin-foam models, the Plebański action turns out in the continuum limit [3, 5].

In this search one important element has been given by gauge formulations. Such formulations have been useful in particular for the incorporation of supersymmetry with gravity to get supergravity [6, 7]. Following this philosophy, MacDowell and Mansouri have written a gauge theory for gravity and for supergravity [8], where the tetrad and the spin connection are components of a gauge field of the group $SO(3, 2)$, respectively of the supergroup $Osp(1|4)$. In this case however, the action contains terms which explicitly break symmetry. Simultaneously to this work, Plebański has formulated a complex BF -action [9], with gauge invariance under $SU(2)$, which can be rewritten as a local Lorentz invariant action with self-dual fields. The relation to gravity arises through the simplicity constraints which act on the two-form B -field, whose solution is given in terms of a tetrad, leading to a self-dual Palatini action, and thus to the Ashtekar action [10]. A supergravity version of this last action, invariant under $SL(2, C)$, has been given by Jacobson [11]. This action has been considered by Capovilla et. al. [12], who, additional to the bosonic constraints, have proposed simplicity constraints on a right handed spinor two-form, whose solution is given in terms of a one-form left-handed spinor. Further, actions invariant under $Osp(2|1)$ have been proposed in [14, 16, 17]. In [14, 15] such actions have been used to study loop quantum supergravity. In [16] an analysis is done for the BF action plus a cosmological term for $N = 1$ and $N = 2$. In [17] the constraints for this action are considered in such a way that the constraints of [12] are written as supertraces under certain irreducible representations of $Osp(2|1)$; in these two works, the Kalb-Ramond on-shell symmetry [27] of the action is explored. Further, in [18] a self-dual version of the MacDowell-Mansouri action of supergravity has been given, with $Osp(4|1)$ gauge fields.

The renormalizability properties of the Plebański action have been studied in [19], where it is argued that the quantum effects can be resummed by an effective, field dependent cosmological constant, leading to a modified “non-metric” formulation. Consequences of this theory have been studied for the Ashtekar formulation, for spin foams in [20], and in cosmology [21]. Another generalization is given in [22], where by relaxing the constraints a generalized potential for the BF theory is proposed, and it is argued that it leads as well to general relativity.

Generalizations of the Plebański action, with real fields, have been studied in [23], where it is extended to higher dimensions, and in [24]. Recently, generalizations have been considered for extended groups in [25], where a Yang-Mills sector arises from the extra gauge fields. More recently, the Plebański action has been traced back to a Matrix theory [26].

In the $OSp(2|1)$ formulation given in [16, 17], the Kalb-Ramond symmetry requires that the Lagrange multipliers satisfy certain additional conditions. Further, in [22] it is shown that as a consequence of the consistency of the equations of motion, the trace of the Lagrange multipliers is constant, the cosmological constant. It could be of interest to see if it works as well for supergravity. We found that both points can be solved by a suitable $OSp(2|1)$ generalization of the action considered in [22], and this is the scope of this work. In Sec. 1, in order to fix notations, we consider the bosonic Plebański theory following Krasnov [22], with unconstrained Lagrange multipliers. Thus the constraint is a mass-like term, and is not given by a trace. In Sec. 2 we consider the $OSp(2|1)$ generalization of this action. It turns out that a straightforward generalization of the constraints of Plebański does not work. In order to avoid this problem, a constraint on the Lagrange multipliers is introduced which, together with the consistency conditions of the equations of motion, is consistent with the supergravity cosmological constant terms and with the Kalb-Ramond invariance of the action. In Sec. 3 we draw some conclusions.

II. PLEBAŃSKI ACTION

In this section we will review the Plebański action and we will try to formulate it in a way suitable for a generalization under an extended symmetry. In particular the fields in the lagrangian should be unconstrained as far as possible. In fact such a formulation has been given in [22]. Let us consider a $SU(2)$ complex one-form connection $\Omega = \Omega_i t^i$ ($i = 1, 2, 3$),

with its two-form field strength $F = d\Omega + \Omega \wedge \Omega$. The generators of the algebra satisfy $[t^i, t^j] = 2i\epsilon^{ij}_k t^k$ and $\text{Tr}(t^i t^j) = 2\delta^{ij}$, where ϵ_{ijk} is the Levi-Civita symbol. The action of Plebański [9] is given by,

$$I = \frac{2i}{k} \int (\text{Tr} B \wedge F + \Phi_{ij} B^i \wedge B^j), \quad (1)$$

where $B = B^i t_i$ is a Lie algebra valued two-form, and the symmetric tensor Φ_{ij} is a zero-form. If we decompose Φ into its irreducible components, its trace $\phi = \text{Tr} \Phi$ and its traceless part $\phi_{ij}^{(t)} = \Phi_{ij} - \frac{1}{3}\phi\delta_{ij}$, we get

$$I = \frac{2i}{k} \int \left(2B^i \wedge F_i + \phi_{ij}^{(t)} B^i \wedge B^j + \frac{1}{3}\phi B^i \wedge B_i \right). \quad (2)$$

The equations of motion of (1) corresponding to the variation of B field and the connection Ω are,

$$F_i + \Phi_{ij} B^j = 0 \text{ and } DB^i = 0. \quad (3)$$

From these equations and the Bianchi identity $DF = 0$, we get the consistency condition

$$D\Phi_{ij} \wedge B^j = 0. \quad (4)$$

Furthermore, a variation with respect to $\phi^{(t)}$, gives the so called simplicity constraints,

$$B^i \wedge B^j - \frac{1}{3}\delta^{ij} B^k \wedge B_k = 0. \quad (5)$$

Frequently, the field ϕ is set to be constant, as done by Plebański [9]. However, as shown in [22], it is not necessary because due to the constraints and the consistency conditions (4), it turns to be constant. Indeed, the conditions (4) can be rewritten as $\epsilon^{\mu\nu\rho\sigma} D_\nu \Phi_{ij} B_{\rho\sigma}^j = 0$, or, contracting them with a B -field,

$$\epsilon^{\mu\nu\rho\sigma} D_\nu \Phi_{ij} B_{\rho\sigma}^i B_{\mu\tau}^j = 0. \quad (6)$$

which can be rewritten as [22] (see Appendix)

$$D_\tau \Phi_{ij} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^i B_{\rho\sigma}^j = 0. \quad (7)$$

Hence, taking into account the constraints (5), $D_\tau \Phi_{ij} \delta^{ij} = \partial_\tau \phi = 0$. Thus, the field ϕ is constant, $\phi = \Lambda$, the cosmological constant.

In order to obtain gravity, the fields in the action (2) are usually decomposed into their real and imaginary parts, $\Omega = \frac{1}{2}(\omega + i\tilde{\omega})$ and $B = \frac{1}{2}(\Sigma + i\tilde{\Sigma})$. Thus we get $F = \frac{1}{2}(R + i\tilde{R})$, where

$$R^i = d\omega^i - \epsilon^i_{jk}\omega^j \wedge \tilde{\omega}^k, \quad (8)$$

$$\tilde{R}^i = d\tilde{\omega}^i + \frac{1}{2}\epsilon^i_{jk}(\omega^j \wedge \omega^k - \tilde{\omega}^j \wedge \tilde{\omega}^k). \quad (9)$$

Further, we define $\omega^{ij} = \epsilon^{ij}_k \tilde{\omega}^k$ and $R^{ij} = \epsilon^{ij}_k \tilde{R}^k$. Then,

$$R^{ij} = d\omega^{ij} + \omega^{il} \wedge \omega_l^j + \omega^i \wedge \omega^j. \quad (10)$$

Thus, if $\omega^{i0} = -\omega^{0i} = -\omega^i$, $\omega^{00} = 0$ and $R^{0i} = R^i$, we get,

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c^b \quad (a, b = 0, 1, 2, 3). \quad (11)$$

The Minkowski metrics we take is $\eta^{ab} = \text{diag}(-1, 1, 1, 1)$. In a similar way, a two tensor Σ^{ab} can be obtained from B^i . Thus, we get the self-dual quantities

$$\Omega^i = \omega^{(+)0i} = \frac{1}{2} \left(\omega^{0i} - \frac{i}{2} \epsilon^{0i}_{cd} \omega^{cd} \right) \quad (12)$$

as well as $B^i = \Sigma^{(+)0i}$ and $R^i = R^{(+)0i}$, which satisfies $R^{(+)}(\omega) = R(\omega^{(+)})$. These self-dual quantities satisfy $\epsilon^{ab}_{cd} \Sigma^{(+)cd} = 2i \Sigma^{(+)ab}$.

Thus, the BF part of the action turns into a self-dual action,

$$I_{BF} = -\frac{i}{k} \int \Sigma^{(+)ab} \wedge R_{ab}^{(+)}. \quad (13)$$

It is well known [9], that the solution of the constraints (5) is given by a factorization of Σ^{ab} as the product of two tetrad real one-forms, and give a basis in the space of the two-forms,

$$\Sigma^{ab} = e^a \wedge e^b. \quad (14)$$

Thus, considering these results and (13), the action (2) can be written as,

$$I = -\frac{1}{4k} \int \left(\frac{1}{2} \epsilon_{abcd} \Sigma^{ab} \wedge \Sigma^{ef} R_{ef}^{cd} d^4x + \frac{\Lambda}{6} \epsilon_{abcd} \Sigma^{ab} \wedge \Sigma^{cd} + 2i \Sigma^{ab} \wedge R_{ab} \right). \quad (15)$$

Further, by means of the Bianchi identity $DT^a = R^a_b \wedge e^b$, where $T^a = De^a$ is the torsion, this action can be written as the sum of the Palatini action with cosmological constant, plus a torsion dependent term,

$$I = -\frac{1}{2k} \int [R(\omega) + 2\Lambda] e d^4x - \frac{i}{4k} \int e^a \wedge DT_a. \quad (16)$$

The action (2) is invariant under the transformations [28],

$$\delta_C B^i = -DC^i, \quad \delta_C \Omega_i = 2\Phi_{ij}C^j, \quad \text{and} \quad \delta_C \Phi_{ij} = 0, \quad (17)$$

where C^i are one-form transformation parameters.

As is well known in the case of pure BF actions [28], the on-shell application of these transformations, corresponds to diffeomorphisms plus field dependent gauge transformations. In our case, if we consider the equations of motion (3), and we write $C_\mu^i = v^\nu B_{\mu\nu}^i$, then [28]

$$\delta_v B_{\mu\nu}^i = v^\rho \partial_\rho B_{\mu\nu}^i + \partial_\mu v^\rho B_{\rho\nu}^i + \partial_\nu v^\rho B_{\mu\rho}^i - \delta_\alpha B_{\mu\nu}^i, \quad (18)$$

$$\delta_v \Omega_\mu^i = v^\nu \partial_\nu \Omega_\mu^i + \partial_\mu v^\nu \Omega_\nu^i - \delta_\alpha \Omega_\mu^i, \quad (19)$$

where δ_α are $SU(2)$ transformations with field dependent parameters $\alpha^i = v^\mu \Omega_\mu^i$. Regarding the corresponding transformations of the fields Φ , we have,

$$\delta_v \Phi_{ij} = v^\mu \partial_\mu \Phi_{ij} - \delta_\alpha \Phi_{ij} = v^\mu D_\mu \Phi_{ij}, \quad (20)$$

which lets the action (2) invariant due to (7).

As have been shown in [10], the self-dual action (13) has also the interesting feature to coincide with the Ashtekar action. (13) has been also obtained in [18] from the MacDowell-Mansouri action.

The action of Plebański corresponds to the embedding of $SO(3)$ into $SU(2)$ given by $v^i \rightarrow v_A^B \sim v_i \sigma_A^i{}^B$, where $\sigma_A^i{}^B$ are the Pauli Matrices. For $SO(3,1)$ the embedding is into $SL(2, C) \otimes \overline{SL(2, C)}$, given for the fundamental representation by $v_a \rightarrow v_{A\dot{B}} = v_a \sigma_{A\dot{B}}^a$, where $\sigma_{0A\dot{B}}$ is the identity matrix and $\sigma_{A\dot{B}}^i$ are the Pauli matrices. With these conventions, the adjoint representation decomposes as,

$$\begin{aligned} \sigma_{A\dot{A}}^a \sigma_{B\dot{B}}^b v_{ab} &= \epsilon_{\dot{A}\dot{B}}(\sigma^{ab}\epsilon)_{AB}v_{ab} + \epsilon_{AB}(\epsilon\bar{\sigma}^{ab})_{\dot{A}\dot{B}}v_{ab} \\ &= -\epsilon_{\dot{A}\dot{B}}v_{AB} + \epsilon_{AB}v_{\dot{A}\dot{B}}, \end{aligned} \quad (21)$$

where $\sigma^{ab} = \frac{1}{4}[\sigma^a, \bar{\sigma}^b]$ and $\bar{\sigma}^{ab} = \frac{1}{4}[\bar{\sigma}^a, \sigma^b]$, satisfy $\epsilon_{cd}^{ab}\sigma^{cd} = 2i\sigma^{ab}$ and $\epsilon_{cd}^{ab}\bar{\sigma}^{cd} = -2i\bar{\sigma}^{ab}$. Hence v_{AB} and $v_{\dot{A}\dot{B}}$ are self-dual, respectively anti-self-dual. Consistently with it, for complex $SU(2)$ vectors we get, $u^i v_i = -\frac{1}{4}u^{(+ab)}v_{ab}^{(+)} = -\frac{1}{4}u^{AB}v_{AB}$.

III. SUPERSYMMETRIC ACTION

The supersymmetric generalization of the Ashtekar formulation has been given and worked out by Jacobson [11], who gave a fermionic $SL(2, C)$ formulation starting from the first order formalism. Further, Capovilla et. al. [12] considered fermionic constraints in addition to the constraints (5). This action has been considered further in [13], where the full supersymmetry invariance is shown, as well as the Kalb-Ramond symmetry which extends (17). In [16] a manifest $OSp(2|1)$ invariant BF action which leads to supergravity was written, with a cosmological term given by the cosmological constant times $S\text{Tr}B^2$, and in [17] the simplicity constraint terms, bosonic plus fermionic, were given as the supertrace under certain irreducible representation of $OSp(2|1)$.

In this section, we give an $OSp(2|1)$ covariant formulation of the Plebański action, following the lines of the previous section, i.e. in the adjoint representation of $OSp(2|1)$, and with the constraints given by a mass-like term. In order that the constraints do not eliminate too many degrees of freedom, a new term which constrains the Lagrange multipliers is required. This term, together with the integrability conditions corresponding to (4), allow the consistency of the action as they ensure, additional to the right constraints, that the cosmological constant terms arise in a similar way as in the bosonic case, and that the action is invariant under the Kalb-Ramond transformations without further conditions on the fields.

The generalization of action (1) is done by making it invariant under the supergroup $OSp(2|1)$, whose algebra is given by $[t_p, t_q] = f_{pq}^r t_r$, where the nonvanishing components of the structure constants are,

$$f_{ij}^k = 2i\epsilon_{ij}^k, \quad f_{iA}^B = \sigma_{iA}^B \text{ and } f_{AB}^i = \sigma_{AB}^i = \epsilon_{BC}\sigma_A^i{}^C. \quad (22)$$

The invariant Killing metric tensor is,

$$\kappa_{pq} = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \epsilon_{AB} \end{pmatrix}, \quad (23)$$

and $S\text{Tr}(T_p T_q) = 2\kappa_{pq}$.

Further, we have $B = B^p t_p = B^i t_i + B^A t_A$, and $\Omega = \Omega^p t_p = \Omega^i t_i + \Omega^A t_A$ ($i = 1, 2, 3$, $A = 1, 2$), where B^A and Ω^A are the fermionic differential forms corresponding to B^i and Ω^i .

The field strength is given by $F = d\Omega + \Omega \wedge \Omega = F^i t_i + F^A t_A$, where,

$$F^i = d\Omega^i + i\epsilon^i_{jk}\Omega^j \wedge \Omega^k - \frac{1}{2}\sigma^i{}_A{}^B\Omega^A \wedge \Omega_B, \quad (24)$$

from which we get,

$$F^{ab} = R^{ab} - \frac{1}{2}\sigma^{ab}{}_A{}^B\Omega^A \wedge \Omega_B, \quad (25)$$

where R^{ab} is given by (11). Furthermore, if we rename $\Omega^A = \sqrt{2k\Lambda}\psi^A$, we have

$$F_A = \sqrt{2k\Lambda} [d\psi_A + \Omega_i \wedge (\sigma^i\psi)_A] = \sqrt{2k\Lambda} D\psi_A, \quad (26)$$

where the covariant derivative can be written also as $D\psi_A = d\psi_A - \frac{1}{2}\omega_{ab} \wedge (\sigma^{ab}\psi)_A$. Here Λ denotes the square root of the cosmological constant of the preceding section, following the usual notation in supergravity.

The natural generalization of the BF part of the Plebański action is $2i \int \frac{1}{2} \text{STr} B \wedge F$, where the supertrace is given by $\text{STr} B \wedge F = \kappa^{pq} B_p \wedge F_q = B^i \wedge F_i + B^A \wedge F_A = B^p \wedge F_p$. However, the straightforward generalization of the constraint term does not work as well. Indeed, it would be given by the supertraceless expression,

$$B^p B^q - \frac{1}{5} \kappa^{pq} B^r B_r = 0. \quad (27)$$

The bosonic part of these constraints is given by $B^i \wedge B^j - \frac{1}{5} \delta^{ij} (B^k \wedge B_k + B^A \wedge B_A) = 0$, from which we get

$$2B^i \wedge B_i = 3B^A \wedge B_A. \quad (28)$$

Substituting this equation into the preceding relation, we get the correct bosonic constraints $B^i \wedge B^i - \frac{1}{3} \delta^{ij} B^k \wedge B_k = 0$, which have the solution (14). However, this solution together with (28) implies that the space-time volume element is fermionic. Moreover, from (27) we get also the constraints $B^i \wedge B^A = 0$, which turn out to be too strong. Indeed, B^i projects on the self-dual part and the solution is given by $B_A = \Sigma^{(-)ab} (\sigma_a \bar{\psi}_b)_A$, which is not consistent with the Rarita-Schwinger action. In fact, the right fermionic constraints were given in [12], $\sigma_{i(AB} B^i \wedge B_C) = 0$.

We propose a manifest $OSp(2|1)$ covariant generalization of (1), with an action given by,

$$I = \frac{2i}{k} \int (\text{STr} B \wedge F + B^p \wedge \Phi_p{}^q B_q + \text{STr} \lambda \Phi), \quad (29)$$

where $\lambda = \lambda^p T_p$ is a four-form field in the adjoint representation, i.e. $(T_p)_q^r = f_{pq}^r$. Thus, the new term in this action is given by

$$\text{Str} \lambda \Phi = \lambda^i (2i\epsilon_i^{jk} \Phi_{jk} - \sigma_i^{AB} \Phi_{AB}) + 2\lambda^A \sigma_A^{iB} \Phi_{iB}. \quad (30)$$

The first two terms on the r.h.s mean that the antisymmetric part of Φ_{ij} and the symmetric part of Φ_{AB} vanish, with no consequences for the constraints. The term which remains, the last one, vanishes in the bosonic limit.

Further, in the same way as in the bosonic case, we have the field equations

$$F_p + 2\Phi_p^q B_q = 0, \quad DB_p = 0, \quad (31)$$

from which follow the consistency conditions

$$D\Phi_p^q \wedge B_q = 0. \quad (32)$$

Similarly to the bosonic case (7) (see Appendix), we get

$$0 = \epsilon^{\mu\nu\rho\sigma} B_{\mu\tau}^p D_\nu \Phi_p^q B_{\rho\sigma q} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^p D_\tau \Phi_p^q B_{\rho\sigma q} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^q B_{\rho\sigma}^p D_\tau \Phi_{pq}. \quad (33)$$

Now, in order to derive the constraints, we decompose the matrix Φ_{pq} into its irreducible components. We have $\Phi^{ij} = \phi^{(t)ij} + \frac{1}{3}\delta^{ij}\Phi_1$ and $\Phi^{AB} = -\frac{1}{2}\epsilon^{AB}\Phi_2$, where $\Phi_1 = \Phi^i_i$ and $\Phi_2 = \Phi^A_A$. Further, $\Phi^{iA} = \Phi^{Ai} = \frac{1}{8}\sigma_{BC}^i \Phi^{BCA} = \frac{1}{8}\sigma_{BC}^i (\phi^{(BCA)} + 2\epsilon^{BA}\phi^C)$, where $\phi^{(BCA)}$ is fully symmetric and $\phi_A = \frac{4}{3}\sigma_A^{iB} \Phi_{iB}$. Thus, up to irrelevant terms, we can write $\text{Str} \lambda \Phi = \frac{3}{2}\lambda^A \phi_A$.

Therefore,

$$I = \frac{2i}{k} \int \left(2B^i \wedge F_i + 2B^A \wedge F_A + \phi^{(t)ij} B_i \wedge B_j + \frac{1}{3}\Phi_1 B^i \wedge B_i - \frac{1}{2}\Phi_2 B^A \wedge B_A \right. \\ \left. + \frac{1}{4}\sigma_{BC}^i \phi^{(BCA)} B_i \wedge B_A + \frac{1}{2}\sigma^{iAB} \phi_B B_i \wedge B_A + \frac{3}{2}\lambda^A \phi_A \right). \quad (34)$$

Hence if we variate with respect to $\phi^{(t)ij}$, we get the constraints (5), the variation of $\phi^{(ABC)}$ gives the fermionic constraint of Capovilla et.al. [12]

$$B_{(A} B_{BC)} = \sigma_{(AB}^i B_{C)} B_i = 0, \quad (35)$$

and the variation of λ^A gives the constraint

$$\phi_A = 0. \quad (36)$$

Therefore, taking into account the decomposition of Φ_{pq} and these constraints, the r.h.s. of (33) gives,

$$\frac{1}{3}D_\tau\Phi_1\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}^iB_{\rho\sigma i}+\frac{1}{2}D_\tau\Phi_2\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}^AB_{\rho\sigma A}=0. \quad (37)$$

This is a linear equation in which the coefficient of $D_\tau\Phi_1$ is bosonic, and the coefficient of $D_\tau\Phi_2$ is fermionic, thus both terms are linear independent as far as the coefficients do not vanish. The first coefficient is the volume element and the second one does not vanish, as will be seen in the following. Therefore we have $D_\tau\Phi_1 = D_\tau\Phi_2 = 0$.

However, considering (36),

$$D_\tau\Phi_1 = \partial_\tau\Phi_1 + 2\sigma_{iAB}\Omega_\tau^A\Phi^{Bi} = \partial_\tau\Phi_1 - \frac{3}{2}\Omega_\tau^A\phi_A = \partial_\tau\Phi_1, \quad (38)$$

$$D_\tau\Phi_2 = \partial_\tau\Phi_2 - 2\sigma_{iAB}\Omega_\tau^A\Phi^{Bi} = \partial_\tau\Phi_2 + \frac{3}{2}\Omega_\tau^A\phi_A = \partial_\tau\Phi_2, \quad (39)$$

Thus $\partial_\tau\Phi_1 = \partial_\tau\Phi_2 = 0$, and we set them to be $\Phi_1 = \Lambda^2$ and $\Phi_2 = 16\Lambda^2$, in such a way that the corresponding terms in (34) have the form of the volume and fermionic mass cosmological constant terms of supergravity [33].

Returning to the constraints, the solution for B_i is given by the self-dual part of (14), and the solution of the fermionic constraint can be obtained making the decomposition $B_A = -\sqrt{k}\Sigma^{ab}(\sigma_{ab}^{BC}\rho_{A(BC)} - \bar{\sigma}_{ab}^{\dot{B}\dot{C}}\rho_{A(\dot{B}\dot{C})})$, from which we get $\phi^{(ABC)}\sigma_{BC}^iB_iB_A = \phi^{(ABC)}\rho_{A(BC)}e^4x=0$, which eliminates the symmetric part of $\rho_{A(BC)}$ and we can write $\rho_{A(BC)} = \frac{1}{2}(\epsilon_{AB}\rho_C + \epsilon_{AC}\rho_B)$. Thus, the self-dual and anti-self-dual parts of B_A are the irreducible components of a one-form spinor field $\bar{\psi}^{\dot{A}}$, which are given by $\bar{\psi}_{A\dot{A}}^{\dot{A}} = -8i\sqrt{\Lambda}\rho_A$ and $\bar{\psi}_{A(\dot{A}\dot{B})} = 8i\sqrt{\Lambda}\rho_{A(\dot{A}\dot{B})}$, in such a way that,

$$B_A = \frac{i}{4}\sqrt{\frac{k}{2\Lambda}}\Sigma^{ab}(\sigma_a\bar{\psi}_b)_A, \quad (40)$$

from which we get

$$B^AB_A = \frac{ik}{8\Lambda}\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b e^4x, \quad (41)$$

which does not vanish.

Thus, the action (34) turns to,

$$I = \frac{2i}{k} \int \left(2B^i \wedge F_i + 2B^A \wedge F_A + \frac{\Lambda^2}{3} B^i \wedge B_i - 8\Lambda^2 B^A \wedge B_A \right). \quad (42)$$

Therefore, taking into account (24), (26) and the solution of the constraints, we get,

$$I = \int \left\{ -\frac{1}{2k} [R(\omega) + 2\Lambda^2] + 2\Lambda (\psi_a \sigma^{ab} \psi_b + \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b) \right\} e d^4x \\ + \int e^a \wedge \bar{\psi}_{\dot{A}} \wedge (\bar{\sigma}_a D\psi)^{\dot{A}} - \frac{i}{4k} \int e^a \wedge DT_a. \quad (43)$$

As usual, a variation of ω in this action gives,

$$\delta_\omega I = \frac{2}{k} \int e^a \wedge \left[T^b - \frac{ik}{2} \psi^A \wedge (\sigma^b \bar{\psi})_A \right] \wedge \delta \omega_{ab}^+ = 0, \quad (44)$$

from which follows the supertorsion vanishing condition $T^a - \frac{ik}{2} \psi^A \wedge (\sigma^a \bar{\psi})_A = 0$. From this condition and from $\psi^A \wedge (\sigma^a \bar{\psi})_A \wedge \psi^B \wedge (\sigma_a \bar{\psi})_B = 0$ follow, modulo surface terms, $\int e^a \wedge DT_a = 0$ and $\int e^a \wedge \bar{\psi}_{\dot{A}} \wedge (\bar{\sigma}_a D\psi)^{\dot{A}} = \int e^a \wedge D\bar{\psi}_{\dot{A}} (\bar{\sigma}_a \psi)^{\dot{A}}$. Therefore we get the supergravity action with cosmological constant [33],

$$I = \int \left\{ -\frac{1}{2k} [R(\omega) + 2\Lambda^2] - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (D_\mu \psi_\nu \sigma_\rho \bar{\psi}_\sigma - \psi_\mu \sigma_\nu D_\rho \bar{\psi}_\sigma) \right. \\ \left. + 2\Lambda (\psi_a \sigma^{ab} \psi_b + \bar{\psi}_a \bar{\sigma}^{ab} \bar{\psi}_b) \right\} e d^4x, \quad (45)$$

Note that the values of the parameters in this action have been chosen in such a way that it is consistent, i.e. that the imaginary part of the final action does not generate new equations of motion, and that the parameters have a physical interpretation. In particular $\text{STr}\Phi = \phi_1 + \phi_2$ is invariant and constant, so we expect naturally that to it corresponds only one scale, given by the cosmological constant.

The transformations corresponding to (17) for the action (29) are easily obtained,

$$\delta_C B^p = -DC^p, \quad \delta_C \Omega_p = 2\Phi_{pq} C^q, \quad \text{and} \quad \delta_C \Phi_{pq} = 0. \quad (46)$$

where C^p are one-form transformation parameters. Similarly to the bosonic case, the on-shell application of these transformations corresponds to diffeomorphisms plus field dependent gauge transformations, giving the same equations as in (18)-(20).

IV. CONCLUSIONS

The formulation of Plebański of gravity continues to be an interesting field of study [19–22, 25, 26]. Its group structure makes it suitable for the study of quantum gravity. One important aspect of it is given by its constraints. In this work we consider its extension

to supergravity, first considering that a straightforward supergroup generalization is inconsistent, because the constraints lead to a fermionic volume element. Thus we consider an $OSp(2|1)$ extension of the action (1), with an additional constraint (29), and leads to the constraints of [12]. Similar to the bosonic case [22], the consistency conditions of the equations of motion, together with this additional constraint, lead to an action which in this case can be identified with the action of supergravity of Townsend [33]. The Kalb-Ramond symmetry of this action is also shown.

As mentioned, the Plebański action has been considered as the starting point for various generalizations as well as quantization proposals for gravity. It would be worth to consider the quantization of the approach presented in this work, in particular for spin-foam models. Similarly for the generalizations proposed in [19–26].

Also the introduction of matter into the action [12] or the generalization to higher dimensions [23] could be considered. The study of the hamiltonian formulation, following e.g. [29], would be also of interest. The consequences of the presence of a cosmological constant regarding the deformation of the symmetry group of discretized models could be also studied [3, 34], as well as the consequences of degenerated metrics [35].

V. APPENDIX

In this appendix we give an algebraic version and a slight generalization of the identity for two forms given in [22], $\iota_\xi B^{(i} \wedge B^{j)} = \frac{1}{2} \iota_\xi (B^{(i} \wedge B^{j)})$. Let us consider the quantity

$$M_{a[b} M_{cd]} = \frac{1}{3} (M_{ab} M_{cd} + M_{ac} M_{db} + M_{ad} M_{bc}). \quad (47)$$

It is easy to show that,

$$M_{a[b} M_{cd]} = -M_{b[c} M_{da]} = M_{c[d} M_{ab]} = -M_{d[a} M_{bc]}. \quad (48)$$

Therefore,

$$M_{a[b} M_{cd]} = M_{[ab} M_{cd]}. \quad (49)$$

All these steps can be repeated if we have a bilinear form with a symmetric Φ_{ij} , and write instead of (47)

$$\Phi_{ij} M_{a[b}^i M_{cd]}^j = \Phi_{ij} M_{[ab}^i M_{cd]}^j. \quad (50)$$

or with an (anti)symmetric Φ_{pq} , $M_{ab}^p M_{cd}^q \Phi_{qp} = M_{cd}^p M_{ab}^q \Phi_{qp}$,

$$M_{a[b}^p M_{cd]}^q \Phi_{qp} = M_{[ab}^p M_{cd]}^q \Phi_{qp}. \quad (51)$$

Thus, in general (in four dimensions, considering a Minkowski signature)

$$M_{a[b}^p M_{cd]}^q \Phi_{qp} = -\frac{1}{24} \epsilon_{abcd} \epsilon^{efgh} M_{e[f}^p M_{gh]}^q \Phi_{qp}. \quad (52)$$

This would work as well with anyonic quantities.

Acknowledgments

We thank O. Obregón, H. García Compeán and G. García for useful discussions. This work was supported in part by CONACyT México Grant 51306 and by BUAP-VIEP grants.

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